PTG-3: Closure-Compatible Metric Selection

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Abstract

Phase Transport Geometry (PTG) requires a metric on the minimal phase manifold $(S^1)^6$ that respects the closure hierarchy established in PTG-1 and the compact transport dynamics introduced in PTG-2. This paper constructs the closure-compatible metric, which projects out closure directions in the triplet and doublet sectors and retains only physically relevant phase differences. The resulting reduced metric defines the primitive curvature scale and provides the geometric backbone for curvature loading, excitation structure, sector organisation (PTG-4), mass scaling (PTG-5), and the physical interpretation developed in the PDT series. A formal uniqueness theorem is stated here and proved in the mathematical closure supplement PDT-MC-1.

This paper is part of the Phase Transport Geometry / Phase Differential Theory series, which develops a unified geometric and operator-theoretic framework for relational transport.

1 Introduction

PTG-1 established the closure hierarchy (1,2,3) and the minimal phase manifold

$$M = (S^1)^6.$$

PTG-2 introduced compact transport dynamics and the curvature operator. PTG-3 now constructs the metric compatible with the closure structure.

Closure directions correspond to collective phase shifts that leave the physical configuration unchanged. A closure-compatible metric must therefore assign zero length to these directions while remaining positive on physically meaningful variations.

2 Closure Directions

The six phase coordinates decompose into:

$$(\theta_1, \theta_2, \theta_3), \qquad (\theta_4, \theta_5), \qquad \theta_6.$$

Each sector contains a closure direction.

Triplet Sector

The closure direction is

$$v_3 = (1, 1, 1)^T.$$

Since closure corresponds to collective phase shifts that do not change the physical configuration, the metric must annihilate this direction. The unique orthogonal projector onto the complementary subspace is therefore

$$P_3 = I_3 - \frac{1}{3} 1_3 1_3^T,$$

where $1_3 = (1, 1, 1)^T$.

Doublet Sector

The closure direction is

$$v_2 = (1, 1)^T.$$

By the same reasoning, the unique orthogonal projector onto the complementary subspace is

$$P_2 = I_2 - \frac{1}{2} 1_2 1_2^T.$$

Singlet Sector

The singlet sector has no closure redundancy. Its metric contribution is simply

$$P_1 = (1).$$

3 The Closure-Compatible Metric

The reduced metric is the block diagonal matrix

$$G_{\text{red}} = \alpha_0 \, \text{diag}(P_3, P_2, P_1),$$

where $\alpha_0 > 0$ sets the primitive curvature scale.

Why block-diagonal?

Block-diagonality follows from sector orthogonality: closure directions in different sectors are independent, and the metric must not mix them. Any non-block-diagonal choice would introduce spurious cross-sector couplings incompatible with the closure hierarchy.

Minimality

Minimality means that the metric removes only the closure directions and nothing more. Any additional projection would eliminate physically meaningful degrees of freedom.

Closure-Compatible Metric Theorem

Among all block-diagonal metrics satisfying:

- closure symmetry,
- sector orthogonality,

- positivity on the reduced space,
- minimality,

the reduced metric

$$G_{\mathrm{red}} = \alpha_0 \,\mathrm{diag}(P_3, P_2, P_1)$$

is unique up to an overall scale.

The full proof is given in PDT-MC-1.

4 Positive Semidefiniteness and the Quotient Space

On the full phase space $(S^1)^6$, the metric is positive semidefinite because closure directions have zero length:

$$G_{\mathrm{red}} v_3 = 0, \qquad G_{\mathrm{red}} v_2 = 0.$$

After quotienting by closure-equivalent configurations, the metric becomes positive definite on the reduced manifold

$$M_{\mathrm{phys}} = (S^1)^6 / \{\text{closure directions}\}.$$

This distinction is important for the geometric interpretation of curvature loading.

5 Interpretation

The closure-compatible metric:

- removes redundant closure directions,
- defines the primitive curvature scale α_0,
- ensures curvature loading is well-defined,
- prepares the ground for excitation structure and mass scaling.

It is the geometric backbone for:

- curvature loading (PTG-2),
- sector structure and structural algebras (PTG-4),
- defect mass scaling (PTG-5),
- finite coherence capacity and the mass gap (PDT-I),
- the mass hierarchy (PDT-II),
- collapse dynamics and the coherence horizon (PDT-III).

6 Conclusion

PTG-3 constructs the closure-compatible metric on $(S^1)^6$. The metric projects out closure directions and defines the primitive curvature scale. This structure underlies curvature loading, sector organisation, mass scaling, and the physical interpretation developed in the PDT series.