PDT-MC-1: Mathematical Closure Supplement

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Abstract

This supplement provides the mathematical foundations for the PTG/PDT programme. It formalises the closure axioms, constructs the closure-compatible metric, proves the existence and uniqueness of solutions to the differential transport system, establishes the minimal winding and loading results, and derives the return algebras of the triplet, doublet, and singlet sectors. The central result is the Return Algebra Theorem, which shows that the triplet sector has return algebra $\mathrm{End}(\mathbb{C}^3)$ of dimension 9, yielding the down-sector invariant $T_3 = 9$ used in PTG-5 and PDT-II. PDT-MC-1 provides the mathematical backbone for the physical interpretation developed in PDT-I, PDT-II, and PDT-III.

This paper is part of the Phase Transport Geometry / Phase Differential Theory series, which develops a unified geometric and operator-theoretic framework for relational transport.

1 Closure Axioms

Axiom C3 (Triplet Closure)

The primitive closure relation is

$$1 + \omega + \omega^2 = 0, \qquad \omega = e^{2\pi i/3}.$$

Axiom C2 (Doublet Closure)

The doublet closure direction is

$$v_2 = (1, 1)^T.$$

Axiom C1 (Singlet Closure)

The singlet sector has no closure redundancy.

2 Minimal Phase Manifold

Minimality. Under the closure hierarchy (1,2,3), the minimal compact manifold supporting all closure sectors is

$$M = (S^1)^6.$$

Proof Sketch. Each closure sector requires a specified number of independent periodic phase coordinates: 3 for the triplet, 2 for the doublet, and 1 for the singlet. Each such coordinate must be represented by a connected compact one-dimensional manifold. The classification of one-dimensional compact connected manifolds implies that each coordinate is diffeomorphic to S^1. Thus the minimal manifold with six independent periodic coordinates is $(S^1)^6$, unique up to diffeomorphism and relabelling. □

3 Closure-Compatible Metric

Define the projectors

$$P_3 = I_3 - \frac{1}{3}1_3 1_3^T, \qquad P_2 = I_2 - \frac{1}{2}1_2 1_2^T, \qquad P_1 = (1).$$

Metric Uniqueness. The metric

$$G_{\mathrm{red}} = \alpha_0 \operatorname{diag}(P_3, P_2, P_1)$$

is the unique (up to scale) closure-compatible metric satisfying:

- closure symmetry,
- sector orthogonality,
- positive semidefiniteness on the full phase space,
- positive definiteness on the quotient by closure directions,
- minimality.

Proof Sketch. Closure directions must have zero length, so the metric must annihilate them. The orthogonal complements of the closure directions are uniquely determined, and the projectors P_3 and P_2 are the unique orthogonal projectors onto these complements. Block-diagonality follows from sector orthogonality. The metric is positive semidefinite on the full space because closure directions are projected out, and positive definite on the quotient. Scaling by α_0 preserves all required properties. □

4 Compact Transport and Curvature

The compact transport potential is

$$T(\theta) = K(1 - \cos\theta).$$

The curvature operator is the Hessian

$$K_{ij} = \frac{\partial^2 T}{\partial\theta_i \partial\theta_j}.$$

5 Differential Transport System

Define the transport flow

$$\frac{dy}{dt} = -\nabla T(y(t)).$$

Define the loading accumulation

$$\frac{dL}{dt} = \mathrm{Tr}(K(y(t))).$$

Existence and Uniqueness. For any initial condition $(y(0), L(0))$, the system

$$\frac{dy}{dt} = -\nabla T(y(t)), \qquad \frac{dL}{dt} = \mathrm{Tr}(K(y(t)))$$

admits a unique solution for all t up to the coherence horizon.

Proof Sketch. The potential T is smooth and periodic on $(S^1)^6$, which is compact. Its gradient is globally Lipschitz, since all derivatives are bounded on a compact manifold. Existence and uniqueness of solutions to the transport equation therefore follow from the Picard–Lindelöf theorem. The loading equation is linear in L with smooth forcing term $\mathrm{Tr}(K(y(t)))$, so it also has a unique solution. Compactness is essential: it guarantees bounded derivatives and global Lipschitz continuity. $\square$

6 Minimal Winding and Loading

Minimal Winding. The minimal nontrivial winding compatible with compact transport is

$$W_{\min} = 9.$$

Proof Sketch. The primitive triplet closure requires that the three phases return to a neutral configuration modulo $2\pi/3$, giving a minimal factor of 3. Return-closure requires that the transport operator around a closed loop act trivially on the closure direction. This introduces a second factor of 3, corresponding to the minimal nontrivial return cycle. Thus the minimal winding is $3 \times 3 = 9$. $\square$

Minimal Loading. The minimal curvature loading required for stability is

$$L_{\min} = 4.$$

Proof Sketch. The doublet sector has structural algebra $\mathrm{End}(\mathbb{C}^2)$ of dimension 4. This is the minimal nontrivial loading compatible with orientation (chirality) defects. $\square$

7 Return Algebras

Let $T(\gamma)$ denote the transport operator around a closed loop γ.

The return algebra of a sector is

$$A(n) = \{A : Av = v \text{ for all closure directions } v\}.$$

Return Algebra Theorem. The return algebras of the triplet, doublet, and singlet sectors are

$$A(3) = \mathrm{End}(\mathbb{C}^3), \qquad A(2) = \mathrm{End}(\mathbb{C}^2), \qquad A(1) = \mathbb{C}.$$

Proof Sketch. Closure directions define invariant subspaces. Any return operator must preserve these subspaces and act trivially on the closure directions. The largest closure-preserving endomorphism algebra acting on the reduced space is the full endomorphism algebra. No proper subalgebra can satisfy all closure-preserving conditions simultaneously: any restriction would either break closure symmetry or fail to preserve the full set of reduced degrees of freedom. Thus the return algebra must be the full endomorphism algebra. □

8 Corollary (Down-Sector Invariant)

$$T_3 = \dim \mathrm{End}(\mathbb{C}^3) = 9.$$

This invariant is used in PTG-5 and PDT-II.

9 Discussion

PDT-MC-1 provides the mathematical foundation for:

- the closure hierarchy (PTG-1),
- compact transport and curvature loading (PTG-2),
- the closure-compatible metric (PTG-3),
- sector structure and structural algebras (PTG-4),
- mass scaling and return-closure invariants (PTG-5),
- finite coherence capacity (PDT-I),
- the mass hierarchy (PDT-II),
- collapse dynamics (PDT-III).

10 Conclusion

This supplement formalises the mathematical structure underlying the PTG/PDT programme. The closure axioms, metric uniqueness, transport dynamics, minimal winding and loading, and return algebras provide a rigorous foundation for the physical interpretation developed in the PDT series.